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Statistics of binomial number fluctuations

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Abstract. When birth terms in the rate equation for a simple birth-death population process are proportional to the difference between the number of individuals present and a larger fixed number, the equilibrium population obeys a binomial distribution. In this paper the statistics of such a process are investigated. Properties of the related counting process, including interval statistics, are calculated and the results used to evaluate the autocorrelation function of a binomial telegraph wave.

1. Introduction

Although the results derived in this paper may find application in a number of areas, the work was stimulated by recent developments in quantum optics and it is appropriate to begin with a brief review of this background motivation [1].

The semiclassical theory of photoelectric detection asserts that the train of detection events registered by a photomultiplier tube constitutes a doubly stochastic Poisson process, i.e. a random train of events whose mean is modulated by the intensity of the classical Maxwell field falling on the detector [2]. This leads naturally to a pre-eminent role for the Poisson and geometric (thermal, Bose-Einstein) distributions in characterising the statistics of photon-counting fluctuations, corresponding to constant and Gaussian field amplitudes, respectively. It further implies that with an ideal detector, (i) the single-interval photon-counting statistics cannot be sub-Poissonian and (ii) the bilinear moment is largest at zero delay time. The quantum theory of photodetection [3], on the other hand, recognises the quantised nature of the Maxwell field and may be interpreted as implying a Bernoulli sampling of the discrete photon flux incident on the detector. Restrictions (i) and (ii) are therefore relaxed, and light giving both sub-Poissonian and antibunched counting statistics can be envisaged. With the recent development of non-classical light sources capable of generating phenomena of this kind [1] there is increasing interest in models which can be used to characterise the sub-Poissonian regime. The purpose of the work presented in this paper is to examine a statistical model generating binomial number fluctuations which span the region between fixed and random number populations.

Binomial states of the radiation field have been studied previously [4-8] but to fully characterise the fluctuations of a train of events it is necessary to define a process which will generate both statistical and correlation properties. Markovian rate equation models for the evolution of photon populations have in the past often provided

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instructive time-dependent descriptions of devices such as classical sources and amplifiers [9–13]. Similar models can be found in the standard statistics literature [14], which provides a useful source of results. However, with a few exceptions [15], sub-Poissonian processes do not appear to be widely discussed. Thus results for the binomial process reported here may well be of interest beyond the quantum optics community and the calculations in later sections will not be linked to specific applications.

In section 2 a simple rate equation model leading to binomial number fluctuations will be discussed. The integrated statistics of a counting process based on emigration from the population will be evaluated in section 3. In section 4 the results will be applied to the calculation of inter-event time interval statistics and the properties of a binomial telegraph wave will be examined. A discussion and concluding remarks follow in section 5.

2. A binomial process

Consider a population with constant death rate μ and birth rate λ. Suppose that losses from the population are directly proportional to the number of individuals present but that the increase in population due to births is proportional to the difference between the population present and a larger fixed number, N. A rate equation defining such a process takes the form

$$\frac{\mathrm{d}P_n}{\mathrm{d}t}=\mu(n+1)P_{n+1}-\mu nP_n-\lambda(N-n)P_n+\lambda(N-n+1)P_{n-1} \tag{1}$$

where $P_n(t)$ is the probability of finding n ($\leqslant N$) individuals present at time t. A generating function for $P_n(t)$ may be defined as follows:

$$Q(s;t)=\sum_{n=0}^{N}(1-s)^n P_n(t) \tag{2}$$

so that the distribution is

$$P_n(t)=\frac{1}{n!}\left(\frac{-\mathrm{d}}{\mathrm{d}s}\right)^n Q(s;t)\big|_{s=1} \tag{3}$$

and its factorial moments are given by

$$N^{[r]}=\langle n(n-1)\ldots(n-r+1)\rangle=\left(\frac{-\mathrm{d}}{\mathrm{d}s}\right)^n Q(s,t)\big|_{s=0}. \tag{4}$$

It is easily verified that Q satisfies the partial differential equation

$$\frac{\partial Q}{\partial t}=-\mu s\frac{\partial Q}{\partial s}-\lambda s(1-s)\frac{\partial Q}{\partial s}-\lambda NsQ. \tag{5}$$

If M ($\leqslant N$) individuals are present initially then the solution of equation (5) governing the population at time t is [16]

$$Q_M(s;t)=[1-(1-\theta)\xi s]^N\left(\frac{1-[(1-\theta)\xi+\theta]s}{1-(1-\theta)\xi s}\right)^M \tag{6}$$

where

$$\theta(t)=\exp[-(\mu+\lambda)t] \tag{7}$$

and

$$\xi = \lambda/(\mu+\lambda). \tag{8}$$

At long times, $t \gg (\mu+\lambda)^{-1}$, an equilibrium is reached with

$$Q_M(s;\infty) = (1-\xi s)^N \tag{9}$$

and the population obeys a binomial distribution:

$$P_n = \begin{cases} {}^N C_n \xi^n (1-\xi)^{N-n} & n \leqslant N \\ 0 & n > N \end{cases} \tag{10}$$

where ${}^N C_n = N!/n!(N-n)!$ are the binomial coefficients. The factorial moments of the equilibrium distribution (10) are given from relation (4) by

$$N^{[r]} = N!\xi^r/(N-r)! \qquad r \leqslant N \tag{11}$$

so that after normalisation

$$n^{[r]} = N^{[r]}/\langle n\rangle^r = (1-N^{-1})(1-2N^{-1})\ldots(1-\overline{r-1}N^{-1}) \tag{12}$$

and the Fano factor, which measures the noise in the population relative to that of a Poisson distribution with the same mean value, takes the form

$$F = (\langle n^2\rangle - \langle n\rangle^2)/\langle n\rangle = 1-\xi. \tag{13}$$

The joint distribution $P_{nn'}$ of finding n individuals present at time t_0 and n' at time t_0+t can be calculated from the generating function

$$\begin{aligned} Q(s,s') &= \sum_{n,n'=0}^{N} (1-s)^n (1-s')^{n'} P_{nn'} \\ &= \sum_{n=0}^{N} (1-s)^n P_n Q_n(s';t) \end{aligned} \tag{14}$$

where Q_n and P_n are given by (6) and (10), respectively, in statistical equilibrium. The sum on the right-hand side of equation (14) can be evaluated exactly to give

$$Q(s,s') = [(1-\xi s)(1-\xi s') + \xi(1-\xi)\theta ss']^N \tag{15}$$

where θ is given by equation (7). The bivariate moment or number fluctuation autocorrelation function may be evaluated using the formula

$$\langle n(0)n(t)\rangle = \left.\frac{\partial^2 Q(s,s')}{\partial s \partial s'}\right|_{s=s'=0} \tag{16}$$

and after normalisation the result obtained is

$$\frac{\langle n(0)n(t)\rangle}{\langle n\rangle^2} = 1 + \frac{1-\xi}{N\xi}\theta(t). \tag{17}$$

Higher-order correlation properties can be derived for the binomial process (1) using a similar approach and it should be possible to derive factorisation theorems of the type which exist for the related birth–death–immigration process and associated Gaussian–Lorentzian intensity fluctuation model [17]. However, these problems will not be addressed here.

The above results will be discussed in section 5 but it is worth noting that the normalised factorial moments are sub-Poissonian and depend only on N, whilst the Fano factor is independent of N being a function only of the parameter ξ defined by equation (8). The fluctuations in the population show positive correlation.

3. The counting process

Various counting methods can be employed to determine the evolution of a population. In classical statistics it is generally assumed that the counting process does not perturb the behaviour of the population, but in the measurement of photon statistics annihilation of individual energy quanta generally occurs, leading to modification of the population process [18]. Fluctuations in a population may also be monitored through measurement of the rate at which individuals leave [18, 19]. This indirect method is relevant to both classical and quantum populations. In the latter case it can be used to describe measurements on a flux of photons leaving a cavity source such as a laser, for example. The train of events generated by monitoring the emigration from a population is a process of interest in its own right in the context of statistical modelling. In this section, therefore, the counting process generated by individuals leaving the population characterised by equation (1) will be investigated.

Suppose, for generality, that the additonal loss rate to be monitored is η and that $P_n(m;\,t)$ is the joint probability that there are n individuals in the population and m have been counted in time t. According to equation (1) the rate equation for the joint distribution is

$$\frac{\mathrm{d}P_n(m;\,t)}{\mathrm{d}t}=\mu(n+1)P_{n+1}(m;\,t)-\lambda(N-n)P_n(m;\,t)+\lambda(N-n+1)P_{n-1}(m;\,t)$$
$$+\eta(n+1)P_{n+1}(m-1,\,t)-\eta nP_n(m;\,t). \tag{18}$$

Defining the associated generating function by

$$Q^c(s,z;\,t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}P_n(m;\,t)(1-s)^n(1-z)^m \tag{19}$$

the required counting distributions follows from the relationship

$$p(m;\,t)=\sum_{n=0}^{\infty}P_n(m;\,t)=\frac{1}{m!}\left(\frac{-\mathrm{d}}{\mathrm{d}z}\right)^m Q^c(0,z;\,t)|_{z=1}. \tag{20}$$

It is not difficult to show that the generating function (19) satisfies the partial differential equation.

$$\frac{\partial Q^c}{\partial t}=-(\mu+\eta)s\frac{\partial Q^c}{\partial s}--\lambda s(1-s)\frac{\partial Q^c}{\partial s}-\lambda NsQ^c+\eta z\frac{\partial Q^c}{\partial s}. \tag{21}$$

A method of solution of this type of equation is outlined in [16] and reproduced in the appendix where it is shown that the equilibrium solution for the counting process of interest here is

$$Q^c(0,z;\,t)=\exp(-N\gamma)\left[\cosh y+\frac{1}{2}\left(\frac{y}{\gamma}+\frac{\gamma}{y}\right)\sinh y\right]^N \tag{22}$$

where

$$\gamma=(\mu+\lambda+\eta)t/2 \tag{23}$$

and

$$y^2=\gamma^2-\eta\lambda t^2 z. \tag{24}$$

Note that if $z\leqslant 1$ then $y^2\geqslant 0$. Formulae (20) and (22) yield

$$\langle m\rangle=\lambda\eta Nt/(\mu+\lambda+\eta) \tag{25}$$

$$\frac{\langle m(m-1)\rangle}{\langle m\rangle^2}=1-\frac{1}{N\gamma}+\frac{1}{2N\gamma^2}-\frac{\mathrm{e}^{-2\gamma}}{2N\gamma^2} \tag{26}$$

$$F = 1 - \frac{2\lambda\eta}{(\mu+\lambda+\eta)^2}\left(1 - \frac{1}{2\gamma} + \frac{e^{-2\gamma}}{2\gamma}\right). \tag{27}$$

These results will be discussed in section 5 but it is evident that in the short-integration time limit, $\gamma \ll 1$, the statistics of the counting process defined by equation (22) reduce to those of the original population, i.e.

$$\lim_{\gamma\to 0} Q^c(0, z; t) = (1 - \langle m\rangle z/N)^N \tag{28}$$

whilst, in general, the number of counts per sample time is not restricted to be less than or equal to N. It is also evident that the Fano factor cannot be less than 0.5, i.e.

$$F \geqslant 1 - 2\lambda\eta/(\mu+\lambda+\eta)^2 \geqslant 1/2. \tag{29}$$

The conditional generating function implied by a rate equation of the type (18) is also given in the appendix of [19] and can be used to calculate the joint counting distribution and hence the correlation properties of the flux of individuals leaving the population. The formulae are cumbersome and only the bilinear moment or autocorrelation function will be quoted here. This takes the form

$$\frac{\langle m(0; t)m(\tau; t)\rangle}{\langle m^2\rangle} = 1 - \frac{\sinh^2\gamma}{N\gamma^2}\exp[-(\mu+\lambda+\eta)t]. \tag{30}$$

The interesting feature of this result is that it shows that fluctuations in the counting process are negatively correlated or antibunched unlike those of the population itself (17) which are bunched.

4. Interval statistics and a telegraph wave model

Other statistical properties of the counting process which can be derived from the results presented above include the distribution of waiting times to the first count [14]

$$p_0(t) = \left.\frac{-\partial Q^c}{\partial t}\right|_{z=1} \tag{31}$$

and the distribution of time intervals between consecutive events [14]

$$p_1(t) = \left.\frac{t}{\langle m\rangle}\frac{\partial^2 Q^c}{\partial t^2}\right|_{z=1}. \tag{32}$$

These two formulae may be evaluated directly from equation (22) and it is convenient to express the results in units of normalised time, γ, given by equation (23). Thus

$$p_0(\gamma) = \tfrac{1}{2}N(1-x^2)\,e^{-N\gamma}[\cosh\gamma x + \tfrac{1}{2}(x+x^{-1})\sinh\gamma x]^{N-1}\left(\frac{\sinh\gamma x}{x} + \cosh\gamma x\right) \tag{33}$$

$$p_1(\gamma) = \tfrac{1}{2}N(1-x^2)\,e^{-N\gamma}[\cosh\gamma x + \tfrac{1}{2}(x+x^{-1})\sinh\gamma x]^{N-2}\left[\left(\cosh\gamma x + \frac{\sinh\gamma x}{x}\right)^2 - \frac{1}{N}\right] \tag{34}$$

where

$$x^2 = 1 - \frac{4\eta\lambda}{(\mu+\lambda+\eta)^2} = 1 - \frac{2\langle m\rangle}{N\gamma} \tag{35}$$

and the average count rate $\langle r\rangle$ is defined by the formula

$$\langle r\rangle=\frac{\langle m\rangle}{t}=\tfrac{1}{2}N(1-x^2). \tag{36}$$

Equations (33) and (34) are plotted in figures 1 and 2 for various values of N and x. The antibunched nature of the counts shows as a displacement of the most probable interval between events away from the origin. This feature is most prominent for small values of N. Setting $N=1$ in equations (33) and (34) leads to

$$p_0(\gamma)=\tfrac{1}{2}(1-x^2)\,\mathrm{e}^{-\gamma}\left(\frac{\sinh\gamma x}{x}+\cosh\gamma x\right) \tag{37}$$

$$p_1(\gamma)=\frac{(1-x^2)}{x}\,\mathrm{e}^{-\gamma}\sinh\gamma x. \tag{38}$$

It is not difficult to establish analytically that $p_0(\gamma)$ is largest at zero time but that $p_1(\gamma)$ displays a maximum when $\gamma=x^{-1}\tanh^{-1}x$.

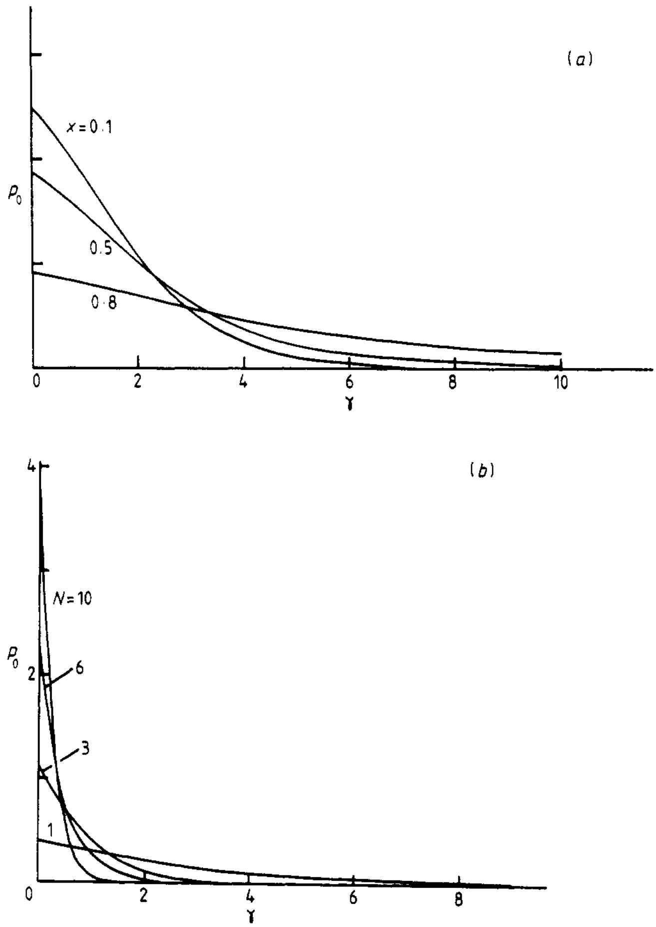

Figure 1. Distribution of times to the first event for a binomial counting process for: (a) $N=1$ and the values of x indicated, (b) $x=0.5$ and the values of N indicated.

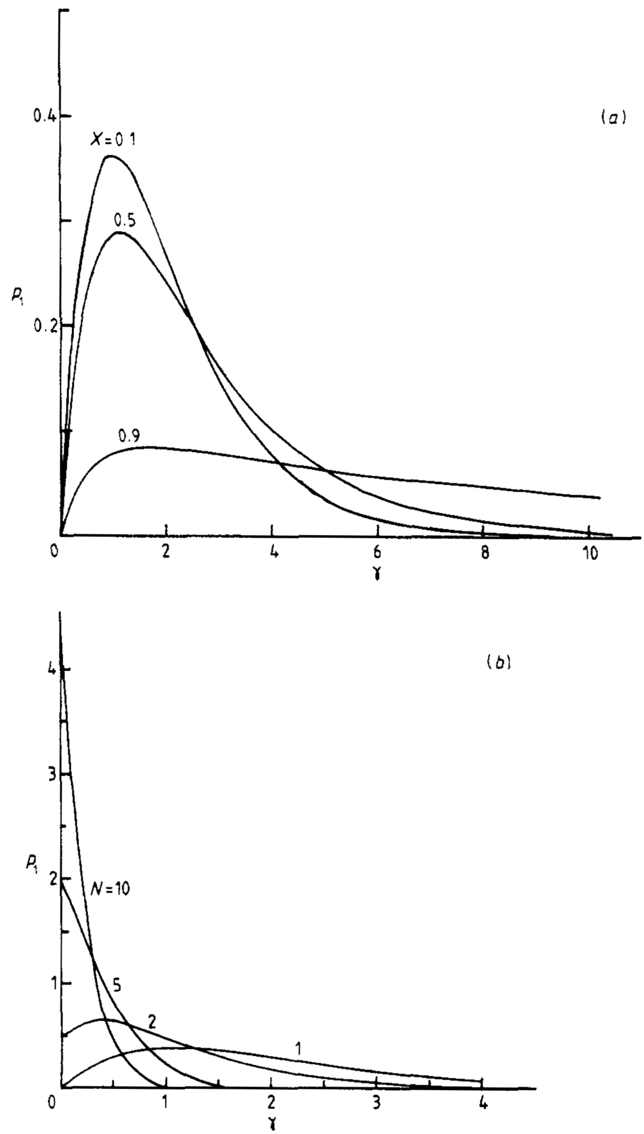

Figure 2. Distribution of times between consecutive events for a binomial counting process for: (a) $N = 1$ and the values of x indicated, (b) $x = 0.5$ and the values of N indicated.

Finally, it is interesting to consider a simple 'continuous' statistical model which can be generated by the counting process investigated in the last section. Suppose that the loss events from the population governed by equation (1) are associated with the zero crossings of a telegraph signal

$$T(t) = \pm 1. \tag{39}$$

Telegraph wave models find application in many areas of science and engineering [20]. It will be assumed for simplicity that T is symmetric

$$\langle T^{2n+1} \rangle = 0 \qquad \langle T^{2n} \rangle = 1 \tag{40}$$

and unbiased. Generalisation to the biased case is possible but will not be considered here [21]. Now, the autocorrelation function of T at two different times t_0, t_0+t is equal to the difference between the probability, P_{even}, of finding an even number of crossings in the interval t and the probability, P_{odd}, of finding an odd number [22]. From equation (19)

$$P_{\text{even}}=\sum_{m=0}^{\infty} P(2m;\, t)=\tfrac{1}{2}[Q^{c}(0,2;\, t)+Q^{c}(0,0;\, t)] \tag{41}$$

$$P_{\text{odd}}=\sum_{m=0}^{\infty} P(2m+1;\, t)=\tfrac{1}{2}[Q^{c}(0,0;\, t)-Q^{c}(0,2;\, t)] \tag{42}$$

so that

$$\langle T(t_0)T(t_0+t)\rangle=Q^{c}(0,2;\, t). \tag{43}$$

Inspection of the solution (22-24) reveals that this quantity can be periodic if $z=2$. It is convenient to express (43) in terms of the parameter

$$\chi^2=1-8\eta\lambda/(\mu+\lambda+\eta)^2. \tag{44}$$

The second term on the right-hand side of this equation is maximised by the choice $\mu=0$, $\lambda=\eta$ and this establishes the inequality

$$-1\leqslant\chi^2\leqslant 1. \tag{45}$$

Thus the autocorrelation function (43) may be written for $t>0$

$$\langle T(0)T(t)\rangle=\begin{cases} e^{-N\gamma}[\cosh\gamma\chi+\frac{1}{2}(\chi+\chi^{-1})\sinh\gamma\chi]^{N} & 0\leqslant\chi^2\leqslant 1\\ e^{-N\gamma}[\cos\gamma|\chi|-\frac{1}{2}(|\chi|-|\chi|^{-1})\sin\gamma|\chi|]^{N} & -1\leqslant\chi^2\leqslant 0\end{cases} \tag{46}$$

where γ is defined by equation (23) as usual. Therefore the autocorrelation function of a binomial telegraph wave can display both monotonic and periodic behaviour depending on the region of parameter space. The simplest periodic structure occurs when $\chi^2=-1$ and $N=1$ leading to

$$\langle T(0)T(t)\rangle=e^{-\gamma}\cos\gamma. \tag{47}$$

Result (46) is plotted for other values of N and χ^2 in figure 3.

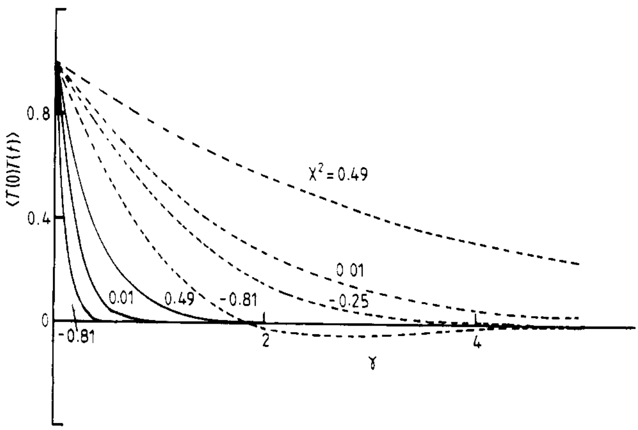

Figure 3. Autocorrelation function of a binomial telegraph wave with the values of χ indicated. $N=1$ (broken curve), $N=10$ (full curve).

5. Discussion of results

The rate equation (1) defines a saturable birth–death process with binomial equilibrium statistics. The solution obtained in section 2 is straightforward and requires little further comment save to remark that the equilibrium distribution, its moments and the Fano factor (13) are the same as would be obtained by Bernoulli sampling a fixed number N of individuals with efficiency ξ. Although the population model (1) and counting process (18) are simply related, the predicted statistical properties are markedly different. As expected, the integrated counting statistics, which can be derived from result (22), reduce to binomial when the integration time is small compared with the population fluctuation time $(\mu+\lambda+\eta)^{-1}$, i.e. $\gamma \ll 1$, but in general $p(m; t)$ does not vanish for $m \geqslant N$. If $\gamma \rightarrow \infty$ with $\langle m \rangle$ held constant then the distribution of m approaches Poisson as fluctuations in the increasingly sparse train of emigrations are averaged out. However, according to equation (27), when transition rates are held fixed, then the smallest value of the Fano factor is achieved in this large-integration time limit. This value is at least one-half, whereas by choosing ξ close to unity in equation (13) the population fluctuations themselves can be made arbitrarily small. The existence of a minimum Fano factor for the counting process is a direct consequence of the random sampling of the population by emigration. As noted in previous work, [18, 19] the normalised autocorrelation function for a counting process is less than that of the monitored population. For the model considered in this paper the reduced correlation leads to an antibunched counting process from a population with positive correlation. Note that the ideal counter assumption, made in section 3, can be relaxed by including a constant detector efficiency factor (multiplying z in equation (24), for example) without substantially changing the results.

The form of the interval distribution, p, shown in figure 2(a) shows that the antibunched character of the detection events is most pronounced at small values of N and x. When $N = 1$ (Bernoulli process [8]) there is at most one individual present in the population at any given time, whilst small x corresponds to a situation in which birth and loss rates are similar with losses occurring only through emigration ($\mu = 0$, $\lambda = \eta$, in equation (35)). This maximises the count rate per coherence (fluctuation) time and ensures the most efficient monitoring of the most antibunched situation. As N increases, the statistics become more Poisson-like and less antibunched (28), (30) and p_0, p, approach negative exponential distributions as shown in figures 1(b) and 2(b). Larger values of x with N fixed correspond to smaller count rates (36) and thus to an increasing likelihood of longer intervals between counts. This leads to the longer tails of p_0, p_1 in figures 1(a) amd 2(a) for the larger values of x.

Plots of the binomial telegraph wave autocorrelation function shown in figure 3 are consistent with the above discussion. A random telegraph wave with Poisson distributed crossings has a negative exponential autocorrelation function [20, 23]. A telegraph wave of equally spaced crossings, on the other hand, has a periodic saw-tooth autocorrelation function [20]. The present model should be close to the former case when N is large but show some periodicity when N is small and x is chosen to optimise both antibunching and monitoring, i.e. $\chi^2 \sim -1$, since $\chi^2 = 1 - 2(1 - x^2)$. Inspection of figure 3 and equation (47) shows that this is indeed the situation. Only the first minimum of the most strongly antibunched situation plotted in figure 3 actually shows up since smaller values of $|\chi|$ (larger values of x) lead to an increase in the expected period relative to the exponential damping whilst the latter always dominates at large values of N. Nevertheless anticorrelation is present for some delays when N is odd and $\chi^2 < 0$ with a smooth change to monotonic behaviour through the marginal case $\chi^2 = 0$.

Finally, although this paper is primarily concerned with calculating basic properties of the statistical model (1) and derivative processes, it is appropriate to comment briefly on applications. In the field of quantum optics, which originally stimulated the work, the binomial process would appear to have potential as a sub-Poissonian input in the modelling of devices such as amplifiers and oscillators. It could also find application in the description of direct transfer devices in which statistical properties are transferred from one set of particles to another during conversion, e.g. electrons to photons in light-emitting diodes [23]. However, applications are obviously not restricted to this specialised field. Telegraph wave models, for example, are widely used by the scientific community and the results of section 4 provide the basis for a new stochastic process of this type.

In conclusion, this paper has examined the properties of a binomial population process and related counting process. It has been shown that although the population is sub-Poissonian, with an arbitrarily small Fano factor, its fluctuations exhibit positive correlation. The counting process, on the other hand, is both sub-Poissonian and antibunched, but the degree to which its noise can be reduced below that of a Poisson train of events with the same mean is limited to a factor of 2. The antibunched character of the counting process is manifest in both distribution of inter-event times and in the correlation properties of a telegraph wave with crossings defined by the events.

Two aspects of the model require further attention: factorisation properties of higher-order correlation functions implied by the Markov nature of the process [17, 25] and the possibilities for numerical simulation. Although it has to be admitted that the model is somewhat artificial at this stage, a number of potential applications can be envisaged both in quantum optics and other areas of science and engineering.

Appendix: solution of equation (21)

Equation (21) must be solved subject to the initial condition

$$Q^c(s, z; t=0) = Q_M(s; \infty) \tag{A1}$$

where Q_M is given by equation (6) of section 2. Normalisation of the equivalent distribution is ensured by the condition.

$$Q^c(0, 0; t) \equiv 1. \tag{A2}$$

The transformation $s = s_0 - k$, $Q^c = Q^0 \exp(\lambda N k t)$ with

$$k = \{[(\mu+\eta+\lambda)^2 - 4\lambda\eta z]^{1/2} - (\mu+\eta+\lambda)\}/2\lambda \tag{A3}$$

reduces equation (21) and condition (A1) to

$$\frac{\partial Q^0}{\partial t} = -s_0\left(\lambda N + (\mu+\eta+2\lambda k+\lambda-\lambda s_0)\frac{\partial}{\partial s_0}\right)Q^0 \tag{A4}$$

$$Q^0(s_0, z; 0) = [1+\lambda(k-s_0)/(\mu+\eta+\lambda)]^N. \tag{A5}$$

The partial differential equation (A4) is identical in form to equation (5) of the text and has solutions of the type (6). A standard method, such as Laplace transformation with respect to the time variable, may be used to incorporate the boundary condition (A5), or alternatively by inspection we find

$$Q^0(s, z; t) = \left[1 + \frac{\lambda}{(\mu+\eta+\lambda)}\left(k - s_0 + \frac{\lambda k s_0(1-\Theta)}{(\mu+\eta+\lambda+2\lambda k)}\right)\right]^N \tag{A6}$$

where

$$\Theta = \exp[-(\mu+\lambda+\eta+2k\lambda)t]. \tag{A7}$$

Defining y and γ as in relations (24) and (23) of the text and noting that we require $Q^c(0, z; t)$, i.e. $s_0 = k$ in equation (A6), it is clear that

$$\begin{aligned} Q^c(0, z; t) &= \exp N(y-\gamma)\{1 + (y-\gamma)^2[1-\exp(-2y)]/4\gamma y\}^N \\ &= \exp(-N\gamma)\left\{\cosh y + \frac{1}{2}\left(\frac{y}{\gamma} + \frac{\gamma}{y}\right)\sinh y\right\}^N. \end{aligned} \tag{A8}$$

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